

Heat transfer in the thermally developing region of a laminar oscillating pipe flow

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A theoretical analysis was performed focusing on the heat transfer associated with the laminar oscillating flow in a tube. This situation finds applications in the Stirling engines or regenerative-type refrigerators where the working fluid in the heat exchanger undergoes an oscillatory motion. Such an oscillating flow conceivably entails the thermally developing region, not only because the swept length of working fluid is roughly equal to or longer than the characteristic length of the heat exchanger, but also because the wall temperature changes abruptly along the longitudinal direction. For simulation of the practical heat exchanger composed of cooler and heater, two types of thermal boundary conditions are taken into account; either wall temperature or wall heat flux has a square-wave distribution. It is found that the thermally developing length increases in proportion to the oscillation frequency at slow oscillation but eventually approaches an asymptotic value at high frequency. The local average Nusselt number in the developing region is observed to be inversely proportional to the square root of the distance measured from the thermal discontinuity. Out of the thermally developing region, the local Nusselt number is determined only by the oscillation frequency regardless of axial position. © 1998 Elsevier Science Ltd. All rights reserved

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Nomenclature

$C_{n,k}$ function of λ defined in Equation (7)
 c_m, d_m complex coefficient
 D diameter of pipe [m]
 $G_{m,n}$ function of η and Λ defined in Equation (11)
 g_k temperature profile function defined in Equation (6)
 I_n first kind of the modified Bessel function of order n
 i imaginary unit number, $\sqrt{-1}$
 J_n first kind of the Bessel function of order n
 L_d thermally developing length [m]
 L_s swept length [m]
 L_w period of the wall temperature distribution [m]
 Nu_0 Nusselt number in case of a sinusoidal wall temperature distribution as $\lambda \rightarrow 0$

Nu_z local Nusselt number
 Nu_λ Nusselt number in case of a sinusoidal wall temperature distribution
 Pr Prandtl number
 q_w dimensionless wall heat flux defined in Equation (14)
 R radius of pipe [m]
 r radial coordinate [m]
 Re Reynolds number
 T temperature [°C]
 T_a amplitude of a sinusoidal wall temperature distribution [°C]
 t time [s]
 u velocity [m/s]
 u_m mean velocity [m/s]
 x dimensionless axial coordinate, $2\pi z/L_w$
 z axial coordinate [m].

Greek letters		ω	angular velocity [rad/s]
β	modified Womersley number, $R\sqrt{\omega/\kappa}$	$ $	absolute value
η	dimensionless radial coordinate, r/R	Real[]	real component of a complex number
κ	thermal diffusivity [m^2/s]	$\langle \rangle$	section averaged quantity, $\langle f \rangle = \frac{1}{2} \int_0^1 f \eta \, d\eta$
Λ	ratio of the swept length to the period of an arbitrary wall temperature distribution	$—$	time averaged quantity, $\bar{f} = \frac{1}{2\pi} \int_0^{2\pi} f \, d\tau$
λ	ratio of the swept length to the period of a sinusoidal wall temperature distribution	$\sim$	locally averaged quantity over the axial
λ_c	critical swept length ratio		direction, $\bar{f}_z = \int_0^z f \, dz/z$
θ	dimensionless temperature, T/T_a	$'$	radial gradient at $\eta = 1$, $f' = df/d\eta(1)$
θ_w	dimensionless wall temperature	sign()	sign of a real number; sign(x) = + 1 for $x > 0$; sign(x) = - 1 for $x < 0$.
$\Delta\theta$	dimensionless temperature difference		
τ	dimensionless time, ωt		

Introduction

The flow pattern in the heat exchanger of a regenerative-type cycle machine such as a Stirling engine/refrigerator and a Vuilleumier refrigerator displays the oscillating features in which the working fluid reverses its flow direction periodically in time. Furthermore, the relevant fluid dynamics and heat transfer phenomena in these system mostly occur in the hydrodynamically and thermally developing regime because the length of the heat exchanger is usually shorter than the swept length of the working fluid and the wall temperature changes abruptly along the longitudinal direction. Besides, the pressure inside the system oscillates as well, due to the variation of the internal volume that the fluid occupies in the machine and/or due to the heat transfer between the fluid and the heat exchanger. Owing to the complexity involved in the phenomena, the heat transfer in this situation has not been analyzed comprehensively considering all these effects and still remained unanswered.

In the early days of research, most of the investigations treated the heat transfer phenomena rather simple way in their analysis; an incompressible oscillating flow with linearly approximated wall temperature distributions¹⁻³. They were successful to reveal the basic features of the heat transfer involved in the oscillating flow with the above assumption. More recently, some researchers have included the compressible effect of the flow field in the analysis^{4,5}. In doing so, they approached the problem with linearization of the governing equations which adopt only the first harmonics and discard all other higher harmonic terms. While the linearization of the governing equations made the problem relatively simple for analytical investigation, this approach could still be readily applicable to the realistic situations where the wall temperature distribution can be linearly approximated or the swept length of the oscillating flow is much shorter than the characteristic length of the heat exchanger. Thus, for example, these investigations have contributed enormously to the better understanding of the heat and work flow especially in the thermoacoustic devices.

However in the heat exchanger of a Stirling cycle

machine, the linearization approach can not be applied since the swept length of the working fluid is in the same order of magnitude as the length of the heat exchanger and furthermore, the wall temperature is not linear anymore but rather changes abruptly along the longitudinal direction⁶. Therefore, the major portion of the heat exchanger pertains to the hydrodynamically and thermally developing region. Consequently, it is necessary to examine the heat transfer within the thermally developing region to properly explain the heat transfer phenomena in the Stirling cycle machine.

Several articles have been published about the developing region of the oscillating flow but are confined to the fluid flow itself^{7,8}. The studies over the attendant heat transfer in the developing region have been few.

Though the flow condition is not exactly the same as the oscillating flow, it is worthy to note the theoretical work of Siegel and Perlmutter⁹ on the heat transfer in the thermally developing region of a pulsating flow between parallel plates. Assuming uniform velocity over the channel cross section, they investigated the explicit dependence of the overall heat transfer on the pulsating frequency. Recently, Kim *et al.*¹⁰ carried out numerical simulations based on a more realistic model and explored the principal mechanism of the heat transfer by the pulsating flow. However, the above-stated works cannot deal with the situations where there exist flow reversals due to the assumptions applied in the analysis. Hence, it is not certain that their results could be extended to the extreme case in which a steady component of the pulsating velocity is zero, i.e. the oscillating flow.

Therefore, our purpose in this paper is to examine theoretically a heretofore unrecognized aspect of the heat transfer process in the thermally developing region by an oscillating flow in a circular pipe. In this interest, the general solution has been found firstly for an arbitrary wall boundary condition by superposing the temperature distributions developed for the sinusoidal wall temperature distributions¹¹. Then, applying the general solution to the situations with the thermal discontinuity along the pipe, the temperature field is obtained in the thermally developing region and the dependency of the thermally developing length on the oscillation frequency is examined. Also, the attendant heat transfer characteristics in the thermally developing region are investigated.

Although this work deals with an incompressible oscil-

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lating flow, we do believe that the current research would contribute to the better understanding of the heat transfer in a Stirling cycle machine since the careful examination has been exercised to accommodate the characteristics of the thermally developing region which has never been investigated yet.

Analysis

Sinusoidal wall temperature distribution

In order to set the scene for the rest of this article, the fundamental case with a sinusoidal wall temperature distribution is adopted from Lee *et al.*¹¹ and summarized. By applying the assumptions of constant thermal properties, negligible axial conduction, and uniform velocity over the cross section with pure sinusoidal oscillation (i.e. $u = L_s \omega \cos(\omega t/2)$), the governing equation for this case is derived as

$$\beta^2 \frac{\partial \theta}{\partial \tau} + 2\beta^2 \lambda \cos \tau \frac{\partial \theta}{\partial x} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right), \quad (1)$$

$$\frac{\partial \theta}{\partial \eta} \Big|_{\eta=0} = 0, \quad \theta \Big|_{\eta=1} = \theta_w = \sin x$$

where the dimensionless variables are

$$\theta = \frac{T}{T_a}, \quad \tau = \omega t, \quad x = 2\pi \frac{z}{L_w}, \quad \eta = \frac{r}{R} \quad (2)$$

and the nondimensional groups are

$$\beta \equiv R \sqrt{\frac{\omega}{\kappa}}, \quad \lambda \equiv \frac{\pi L_s}{2L_w}. \quad (3)$$

In the above equations, T_a and L_w are the amplitude and the period of a sinusoidal wall temperature distribution respectively and L_s is the swept length of the oscillating flow defined as

$$L_s \equiv \frac{1}{2} \int_0^{2\pi} |u| d\tau. \quad (4)$$

The two nondimensional numbers, i.e. β and λ represent respectively a ratio of the radius to the thickness of the Stokes' thermal boundary layer and a ratio of the swept length to the characteristic length of the wall temperature distribution.

The solution to Equation (1) is¹¹

$$\begin{aligned} \theta = \sin x + \text{Real} \left[\sum_{k=1}^{\infty} C_{0,k}(\lambda) g_k \sin x \right. \\ \left. + \sum_{n=1}^{\infty} \left\{ \sum_{k=-\infty, k \neq 0}^{\infty} C_{2n,k}(\lambda) g_k \exp(2ni\tau) \sin x \right. \right. \\ \left. \left. + i \sum_{k=-\infty, k \neq 0}^{\infty} C_{2n-1,k}(\lambda) g_k \exp((2n-1)i\tau) \cos x \right\} \right] \end{aligned} \quad (5)$$

in which

$$g_k \equiv 1 - \frac{I_0(\beta \sqrt{ik} \eta)}{I_0(\beta \sqrt{ik})} \quad (6)$$

and

$$C_{n,k}(\lambda) \equiv (-1)^{k+1} J_k(2\lambda) J_{n-k}(2\lambda). \quad (7)$$

Arbitrary wall boundary condition

In this section, we construct a solution corresponding to a more complicated wall temperature distribution from the fundamental temperature distribution, Equation (5), with the aid of superposition principle. In this solution procedure, the assumptions introduced to derive the fundamental temperature distribution, such as a uniform velocity assumption, are consistently effective to the general solution. Though the uniform velocity profile corresponds to a limiting case of $\text{Pr} \ll 1$, the heat transfer characteristics in this case do not deviate much from those for a moderate value of Pr , thereby the solution with uniform velocity assumption can successfully predict the heat transfer in real situations with 2D velocity profile¹¹.

In what follows, the general solutions are developed for two kinds of wall boundary condition, i.e. arbitrary wall temperature distribution and arbitrary wall heat flux distribution.

Arbitrary wall temperature distribution. The arbitrary wall temperature distribution, whenever it is periodic, can be expanded in a Fourier series

$$\theta_w = \sum_{m=-\infty}^{\infty} c_m \exp(imx) \quad (8)$$

in which

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_w \exp(imx) dx. \quad (9)$$

Since the fluid temperature corresponding to each term in Equation (8) can be obtained directly from Equation (5), the temperature field subject to the thermal boundary condition of Equation (8) can be expressed as

$$\theta = \theta_w + \text{Real} \left[\sum_{m=-\infty}^{\infty} c_m \exp(imx) \sum_{n=0}^{\infty} G_{m,n}(\eta, \Lambda) \exp(in\tau) \right] \quad (10)$$

where $G_{m,n}$ is

$$G_{m,n}(\eta, \Lambda) \equiv \begin{cases} \sum_{k=1}^{\infty} C_{0,k}(m\Lambda) g_k(\eta), & n=0 \\ (-1)^n \sum_{k=-\infty, k \neq 0}^{\infty} C_{n,k}(m\Lambda) g_k(\eta), & n>0 \end{cases} \quad (11)$$

in which Λ is the ratio of the swept length to the period

of wall temperature distribution of Equation (8) and is written by

$$\Lambda = \frac{\pi L_s}{2L_w}. \tag{12}$$

From Equation (10), the time-averaged fluid temperature distribution is reduced to

$$\bar{\theta} = \theta_w + \text{Real} \left[\sum_{m=-\infty}^{\infty} c_m G_{m,0}(\eta, \Lambda) \exp(imx) \right] \tag{13}$$

where the bar denotes a time-averaged value.

The wall heat flux is obtained by differentiating Equation (10) and evaluating it at $\eta = 1$. The instantaneous wall heat flux is then written as

$$q_w = \frac{\partial \theta}{\partial \eta} \Big|_{\eta=1} = \text{Real} \left[\sum_{m=-\infty}^{\infty} c_m \exp(imx) \sum_{n=0}^{\infty} G'_{m,n} \exp(in\tau) \right] \tag{14}$$

and the time-averaged value is

$$\bar{q}_w = \text{Real} \left[\sum_{m=-\infty}^{\infty} c_m G'_{m,0} \exp(imx) \right] \tag{15}$$

where $G'_{m,n}$ is defined as

$$G'_{m,n} \equiv \frac{\partial G_{m,n}}{\partial \eta} \Big|_{\eta=1}. \tag{16}$$

Arbitrary wall heat flux distribution. When the wall heat flux distribution is given as a boundary condition, the given heat flux is usually constant in time. However, if the heat capacity of the wall is not zero (i.e. the wall thickness is finite), the heat flux at the inner wall varies with time in a period due to the effect of the wall absorbing heat from and rejecting it to the fluid instantaneously. In other words, the given heat rate as the boundary condition is the value at the outer wall and the heat flux at the inner wall oscillates with time, though its time-averaged value is equal to the heat flux at the outer wall.

Therefore, the boundary condition could be given as the time-averaged heat flux at the inner wall and may be expressed as

$$\bar{q}_w = \sum_{m=-\infty}^{\infty} d_m \exp(imx), d_0 = 0. \tag{17}$$

In the above, d_0 , which is the average heat flux over the axial direction, must be zero for the temperature field to achieve a periodic steady state.

When the fluid temperature in this case is expressed in the form of Equation (10), the time-averaged wall heat flux must satisfy Equation (15). Therefore, a relation between the Fourier coefficients for the wall heat flux and those for the wall temperature can be obtained by comparing Equations (15) and (17). Substitution of this relation into Equation (10) yields the temperature field with given wall heat flux distribution as follows:

$$\theta = \theta_w + \sum_{m=-\infty}^{\infty} \frac{1}{\text{Real}[G'_{m,n}]} \text{Real} \left[d_m \exp(imx) \sum_{n=0}^{\infty} G_{m,n} \exp(in\tau) \right] \tag{18}$$

where the wall temperature θ_w is

$$\theta_w = \sum_{m=-\infty}^{\infty} \frac{d_m}{\text{Real}[G'_{m,0}]} \exp(imx) \tag{19}$$

Thermally developing region

Using the foregoing results, we can examine how the heat transfer in an oscillating pipe flow is influenced by the sudden change of the wall boundary condition along the axis.

As mentioned earlier, the thermally developing region occurs due to the abrupt change of the boundary condition. However, a stepwise variation along the axis is not spatially periodic and therefore we cannot directly utilize the foregoing solutions. Under the situation where the swept length of the fluid is much shorter than the characteristic length of the wall boundary condition, i.e.

$$\Lambda \ll 1 \tag{20}$$

the square-wave distribution is expected to yield almost the same features of heat transfer as in the stepwise distribution. This is due to the fact that the neighboring periods of distributions have insignificant influence on the heat transfer of the region to be investigated.

In what follows, we consider two square-wave thermal forcings (either temperature or heat flux; as shown in Figure 1) and derive the relevant heat transfer characteristics by means of the foregoing generalized solutions. The heat is supplied or removed at each plateau in Figure 1 and thus each plateau can be physically interpreted as a heater or a cooler. The heater and cooler can be represented with two different types of boundary conditions; first, the heater and cooler maintain the constant wall temperatures in the case of temperature forcing boundary condition. Second, they supply constant wall heat fluxes in the case of heat flux forcing boundary condition. Also, it can be assumed that there exists an imaginary piston on the center of each part to generate the oscillating flow.

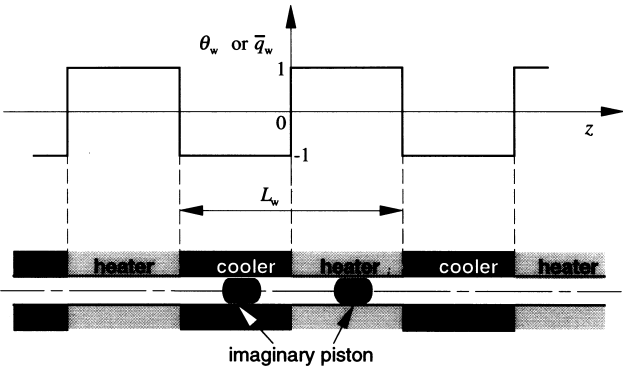


Figure 1 Square-wave-form distribution of the wall boundary condition

Temperature forcing

In this case, the wall temperature is expressed as a Fourier series

$$\theta_w = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin(2m-1)x \quad (21)$$

and then the fluid temperature can be obtained from Equation (10) as

$$\begin{aligned} \theta = \theta_w + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \operatorname{Real}[G_{2m-1,0} \sin(2m-1)x \\ + \sum_{n=1}^{\infty} \{G_{2m-1,2n} \exp(2ni\tau) \sin(2m-1)x \\ - iG_{2m-1,2n-1} \exp((2n-1)i\tau) \cos(2m-1)x\}]. \end{aligned} \quad (22)$$

Evaluating from Equation (22), the time evolution of the fluid temperature within a period of a single oscillation is shown as series of 3D plots in Figure 2. The interval of

time sequence is selected as 0.5π . The first frame of a series corresponds to the moment when the fluid arrives at the left most side within its cyclic motion and is about to reverse its flow direction. The longitudinal coordinate is normalized by the swept distance L_s instead of L_w . With this representation, the temperature distribution near the thermal discontinuity is assured to be independent of the period of wall boundary condition as long as $\Lambda \ll 1$. In other words, the square-wave boundary condition would yield the same heat transfer characteristics with the stepwise boundary condition.

Considering the case of $\beta = 1$ in Figure 2, it is found that the transverse fluid temperature distribution is parabolic and the longitudinal distribution is similar to that of wall temperature though it varies a little with time. As β increases, however, the longitudinal temperature distribution of the fluid near the center of the pipe starts to deviate from that of the wall temperature and eventually becomes a constant profile which is invariant with time and reciprocates as if it is a solid body. This is because at large β , the heat transfer from the wall does not penetrate into the core part but is limited only within a thin annular space near the wall. Meanwhile, the results obtained in this study

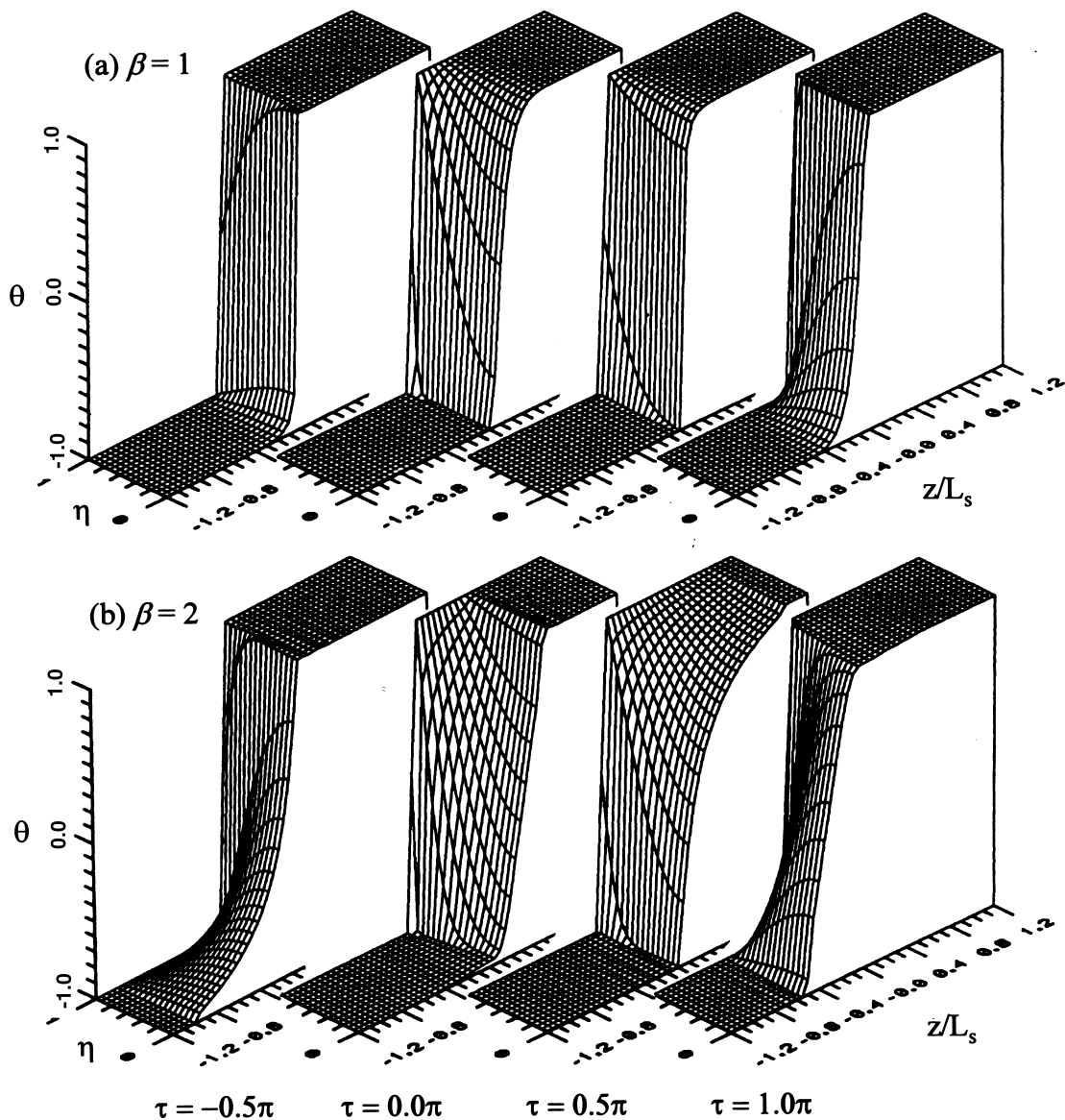


Figure 2 Timewise variations of fluid temperature near the wall temperature discontinuity

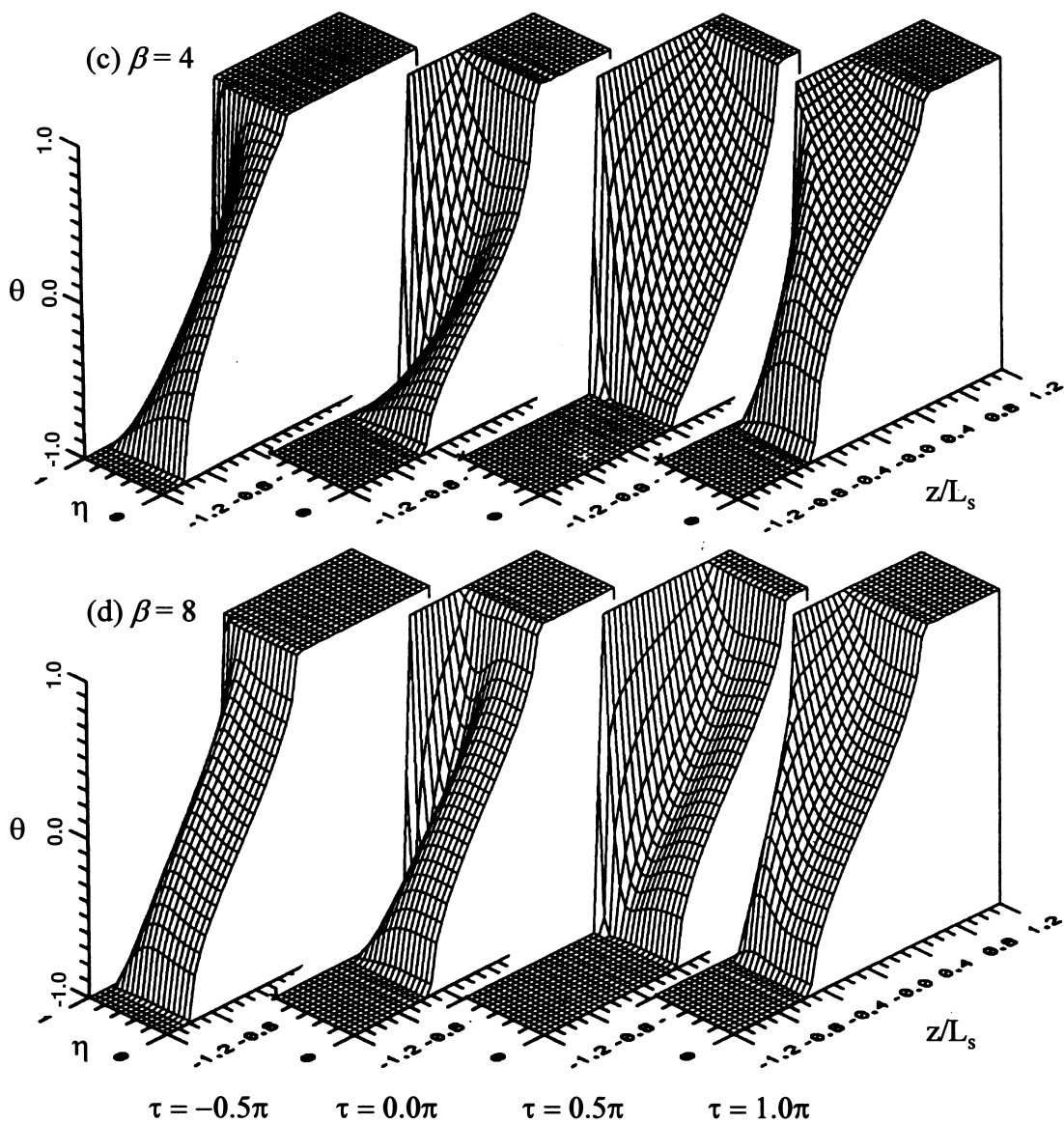


Figure 2 Timewise variations of fluid temperature near the wall temperature discontinuity

such that the longitudinal temperature distribution of the fluid is completely different from that of the wall at large β , cannot be obtained from the analysis with linealized equations and show the distinguished results of the present work from others.

The section-time-averaged temperature of the fluid, $\langle \bar{\theta} \rangle$ is shown in Figure 3 to clarify the region affected by the thermal discontinuity.

The thermally developing length may be defined as the distance at which the temperature difference between the wall and the fluid becomes zero. Then, it can be found from Figures 2 and 3 that the thermally developing length increases with the oscillation frequency but converges to the swept length of the fluid if the frequency parameter β becomes approximately greater than 2. The reason why the thermally developing length becomes equal to the swept length is that the fluid particles which oscillate in the region $z > L_s$ cannot ever experience the thermal discontinuity (note that the axial diffusion is neglected in the present study). Consequently, the effect of the change of the wall temperature can expand no farther than the swept length. The above discussion is relevant to the result of Iguchi *et al.*⁸ that, when the oscillation frequency is high enough, the

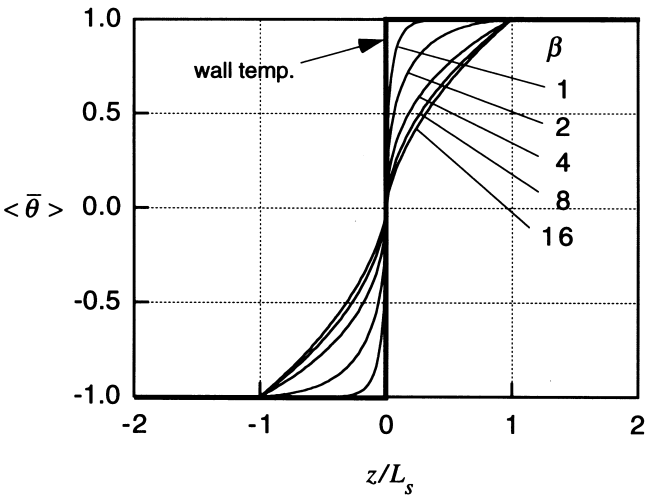


Figure 3 Section-time-averaged fluid temperature in the thermally developing region in case of temperature forcing

length needed for the oscillating velocity to achieve a fully developed profile is also equal to the swept length of the fluid.

Heat flux forcing

When the wall heat flux is distributed in a square-wave form, the fluid temperature can be obtained by Equation (18) similarly to the case of temperature forcing. Unlike the case of the temperature forcing, however, neither θ_w nor $\langle \bar{\theta} \rangle$ is independent on Λ over the entire domain. This precludes the identification of the thermally developing region directly from the temperature distribution of the wall or the fluid. Fortunately, it is recognized that the time-averaged temperature difference

$$\langle \Delta \bar{\theta} \rangle \equiv \theta_w - \langle \bar{\theta} \rangle \quad (23)$$

$$= -\frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\text{Real}[\langle G_{2m-1,0} \rangle]}{(2m-1)\text{Real}[G'_{2m-1,0}]} \sin(2m-1)x$$

is independent of the swept length ratio and thus can be utilized to define the thermally developing region. The axial distribution of $\langle \Delta \bar{\theta} \rangle$ for several values of β is drawn in Figure 4. It is evident that there exists an asymptotic value of $\langle \Delta \bar{\theta} \rangle$ for each β , thereby implying the existence of a fully developed region at large z/L_s . Meanwhile, the variation of $\langle \Delta \bar{\theta} \rangle$ at small z/L_s indicates the presence of the thermally developing region.

The asymptotic value, $\langle \Delta \bar{\theta} \rangle_{\text{asympt}}$, can be found by transforming the summation in Equation (23) into an integral and examining the asymptotic behavior of the integral as z/L_s goes infinite.

Firstly, Equation (23) can be rewritten with the definition of $Nu_{m\Lambda}$

$$Nu_{m\Lambda} \equiv -2 \frac{\text{Real}[G'_{m,0}]}{\text{Real}[\langle G_{m,0} \rangle]} \quad (24)$$

such as

$$\langle \Delta \bar{\theta} \rangle = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)x}{(2m-1)Nu_{(2m-1)\Lambda}} \quad (25)$$

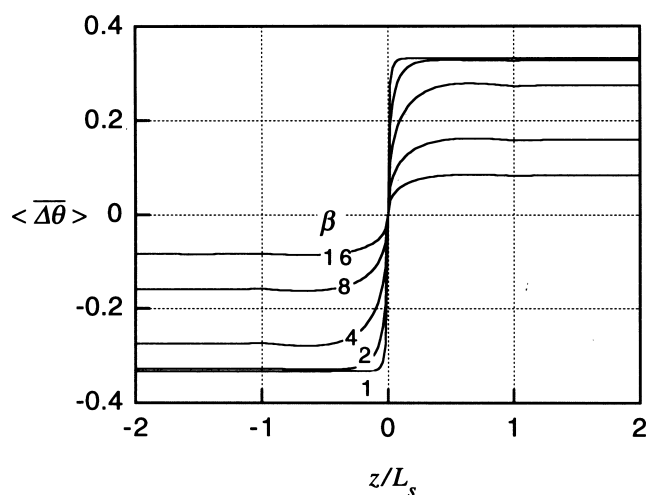


Figure 4 Section-time-averaged difference between wall and fluid temperatures in the thermally developing region in case of heat flux forcing

$Nu_{m\Lambda}$ implies the Nusselt number in case of a sinusoidal wall temperature distribution with $\lambda = m\Lambda$ and the characteristics of this quantity are well described by Lee *et al.*¹¹.

Since we are dealing with the case of $\Lambda \ll 1$ [see Equation (20)], Equation (25) can be transformed into an integral by letting $(2m-1)\Lambda = \lambda$ and $2\Lambda = d\lambda$, such that

$$\langle \Delta \bar{\theta} \rangle = \frac{4}{\pi} \int_0^{\infty} \sin\left(4\lambda \frac{z}{L_s}\right) / (\lambda Nu_{\lambda}) d\lambda \quad (26)$$

In order to perform the integration and thus to find the asymptotic value of $\langle \Delta \bar{\theta} \rangle$ as $z/L_s \rightarrow \infty$, the functional relationship of Nu_{λ} upon λ is needed. With the reference to Lee *et al.*¹¹, it can be found that Nu_{λ} is dependent only on β when λ is small and becomes to increase as λ increases over a certain value, i.e.

$$Nu_{\lambda} \approx \begin{cases} Nu_0, Nu_0 \equiv \lim_{\lambda \rightarrow 0} Nu_{\lambda} & \text{for } \lambda \ll \lambda_c \\ 1.52\beta\sqrt{\lambda} & \text{for } \lambda \gg \lambda_c \end{cases} \quad (27)$$

in which Nu_0 represents the Nusselt number for the infinitesimally small λ and is a function of β only. And λ_c implies the critical value of λ above which the dependency of Nu_{λ} upon λ changes. The parameter λ_c can be quantified as the value of λ at the intersecting point of the two extreme curves of Nu_{λ} , i.e.

$$\lambda_c \equiv \left(\frac{Nu_0}{1.52\beta} \right)^2 \quad (28)$$

Since Nu_0 is a function of β only, λ_c also depends only on β . Nu_0 and λ_c are drawn from Lee *et al.*¹¹ and shown in Figure 5. It can be found from the figure that there also exist two asymptotes for both of Nu_0 and λ_c as below:

$$Nu_0 \approx \begin{cases} 6 & \text{as } \beta \rightarrow 0 \\ \sqrt{2}\beta & \text{as } \beta \rightarrow \infty \end{cases} \quad (29)$$

$$\lambda_c \approx \begin{cases} \frac{36}{1.52^2} \beta^{-2} & \text{as } \beta \rightarrow 0 \\ \frac{2}{1.52^2} & \text{as } \beta \rightarrow \infty \end{cases} \quad (30)$$

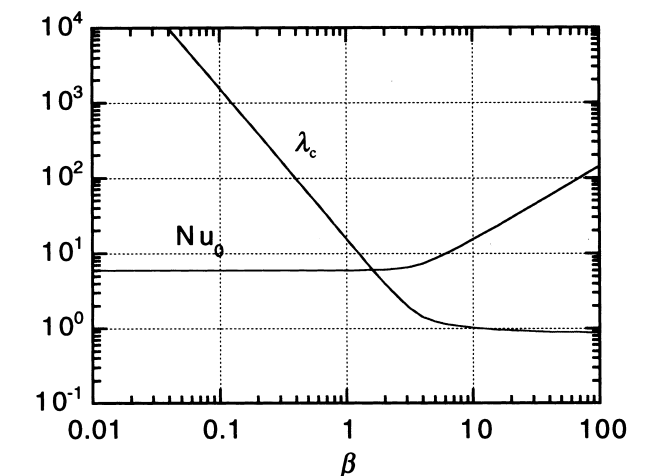


Figure 5 Variation of Nu_0 and λ_c with respect to the modified Womersley number

With the aid of Nu_0 and λ_c , the Nu_λ curves are normalized and depicted in Figure 6. This figure reveals that all the Nu_λ/Nu_0 curves for different values of β have almost same dependency on λ/λ_c regardless of β . It is also found that the normalized curves have two asymptotes:

$$\frac{Nu_\lambda}{Nu_0} \approx \begin{cases} 1 & \text{as } \lambda/\lambda_c \rightarrow 0 \\ \sqrt{\lambda/\lambda_c} & \text{as } \lambda/\lambda_c \rightarrow \infty \end{cases} \quad (31)$$

Using this relation, Equation (26) can be approximated as

$$\begin{aligned} \langle \Delta \theta \rangle &= \frac{4}{\pi} \frac{1}{Nu_0} \int_0^\infty \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \frac{Nu_\lambda}{Nu_0} \right) d \left(\frac{\lambda}{\lambda_c} \right) \\ &\equiv \frac{4}{\pi} \frac{1}{Nu_0} \left\{ \int_0^1 \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \right) d \left(\frac{\lambda}{\lambda_c} \right) + \int_1^\infty \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \right)^{\frac{3}{2}} d \left(\frac{\lambda}{\lambda_c} \right) \right\}. \end{aligned} \quad (32)$$

In order to find the asymptotic value of $\langle \Delta \theta \rangle$, we examine the behavior of the integrals in Equation (32) when z/L_s goes infinite. When $z/L_s \gg 1$, Equation (32) can be simplified as¹²

$$\begin{aligned} \langle \Delta \theta \rangle &\equiv \frac{4}{\pi} \frac{1}{Nu_0} \left\{ \int_0^\infty \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \right) d \left(\frac{\lambda}{\lambda_c} \right) \right. \\ &\quad \left. - \int_1^\infty \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \right) d \left(\frac{\lambda}{\lambda_c} \right) + \int_1^\infty \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \right)^{\frac{3}{2}} d \left(\frac{\lambda}{\lambda_c} \right) \right\} \\ &\equiv \frac{4}{\pi} \frac{1}{Nu_0} \left\{ \int_0^\infty \sin \left(4\lambda_c \frac{\lambda}{\lambda_c} \frac{z}{L_s} \right) / \left(\frac{\lambda}{\lambda_c} \right) d \left(\frac{\lambda}{\lambda_c} \right) + O \left(\left(\lambda_c \frac{z}{L_s} \right)^{-2} \right) \right\} \\ &= \frac{2}{Nu_0} \left\{ \text{sign}(z) + O \left(\left(\lambda_c \frac{z}{L_s} \right)^{-2} \right) \right\}. \end{aligned} \quad (33)$$

Remembering that we are dealing with the case of a square-wave distribution of the wall heat flux with the condition of $\Lambda \ll 1$, i.e. the stepwise distribution, Equation (33) can be rewritten as

$$\langle \Delta \theta \rangle_{\text{asympt}} = \frac{2}{Nu_0} \bar{q}_w. \quad (34)$$

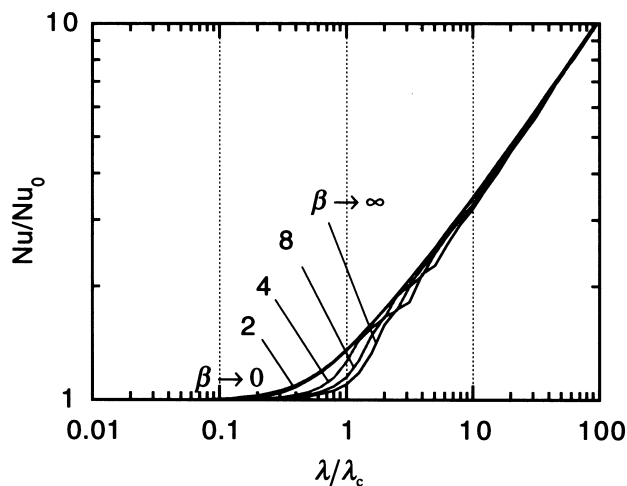


Figure 6 Variation of Nu/Nu_0 with respect to λ/λ_c

Here, worthy of remark is the appearance of Nu_0 in Equation (34). This dictates that the heat transfer coefficient in the thermally developed region for the heat flux forcing is identical to that corresponding to a sinusoidal temperature distribution with small λ . Such a similarity for the asymptotic behavior of $\langle \Delta \theta \rangle$ may arise from the dominance of a long-wave-length sinusoidal component over the thermally developed region.

Figure 4 reveals that the axial position at which $\langle \Delta \theta \rangle$ achieves its asymptotic value (i.e. the characteristic length of the thermally developing region) increases with β for small β and tends to converge to a constant but apparently order of L_s . Although this situation is very similar to the case of temperature forcing, a quantitative determination of the thermally developing length requires an additional manipulation due to the tricky behavior of $\langle \Delta \theta \rangle$ near the transition point.

The thermally developing length

Our attempt to quantify the thermally developing length begins with introducing a new characteristic length in such a manner that all the curves in Figure 4 fall onto a single curve, i.e. a normalized curve. A clue to the new length scale can be obtained from Equation (32). By choosing $\lambda_c z/L_s$ as a new independent variable instead of z/L_s , the integrals in Equation (32) become independent of β and thus the temperature difference curves can be normalized.

Figure 7 shows the reconstructed plot of Figure 4 adopting $\lambda_c z/L_s$ and $Nu_0 \langle \Delta \theta \rangle$ as new coordinates. It can be found from this figure that all the $Nu_0 \langle \Delta \theta \rangle$ curves for each β coincide nearly with each other, though the profiles are a little different. It is also found that the curves converge to an asymptotic value all together at the vicinity of $\lambda_c z/L_s = 1$ with a maximum deviation of 2%. This reveals that the developing region is confined in the region of $\lambda_c z/L_s < 1$. Therefore, the length of the developing region can be prescribed as

$$L_d = \frac{1}{\lambda_c} L_s. \quad (35)$$

Since λ_c depends on β , L_d is also a function of β . The variation of the thermally developing length with respect

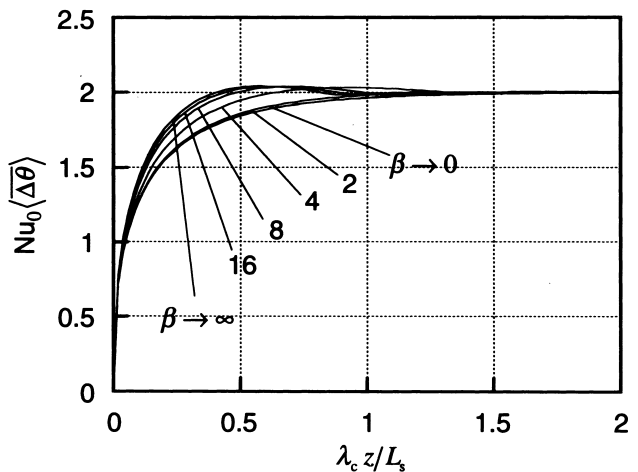


Figure 7 Normalized distributions of the temperature difference in the thermally developing region in case of heat flux forcing

to β can be obtained by Equation (30) or Figure 5 which describes the dependency of λ_c upon β .

In conclusion, the asymptotic behavior of the thermally developing length L_d can be described as

$$L_d/L_s \approx \begin{cases} \frac{1.52^2}{36} \beta^2 & \text{as } \beta \rightarrow 0 \\ \frac{1.52^2}{2} & \text{as } \beta \rightarrow \infty \end{cases} \quad (36)$$

The variation of L_d/L_s together with the asymptotic curves from Equation (36) is depicted in Figure 8. It is interesting to note that the asymptotic curve for $\beta \rightarrow 0$ is practically valid also for moderate values of β and the same is true for the upper limiting case of $\beta \rightarrow \infty$. In practical applications, the normalized frequency β is an order of 10 for both heaters and coolers of Stirling cycle machines. The results in Figure 8 indicate that the thermally developing length of the heater and cooler is nearly equal to the swept length of the working fluid. By contrast, the normalized frequency β is typically below 1 for the regenerator. Although our analysis does not account for the presence of regenerator, it can be inferred that the thermally developing length of a regenerator is proportional to β^2 . In view of practical applications, our analysis implies that the effect of the thermally developing region must be considered for the heater and cooler but it is negligible in regenerator due to a large disparity in the relevant β value.

It is also worthy to compare the thermally developing length between the oscillating flow and the unidirectional flow. We consider a low frequency limit, i.e. $\beta \rightarrow 0$, since the oscillating flow can be thought to have features similar to the unidirectional steady flow when the oscillation frequency is extremely low.

It is well known that the thermally developing length for the unidirectional laminar flow in a tube (supposing slug flow and constant heat flux with 2% deviation) is approximately $L_d/D \cong 0.058 \text{Re Pr}^{1/3}$. If the mean velocity of the oscillating flow

$$u_m = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \frac{L_s \omega}{2} |\cos \omega t| dt = L_s \omega / \pi \quad (37)$$

is used as a velocity scale in $L_d/D \cong 0.058 \text{Re Pr}$, we have

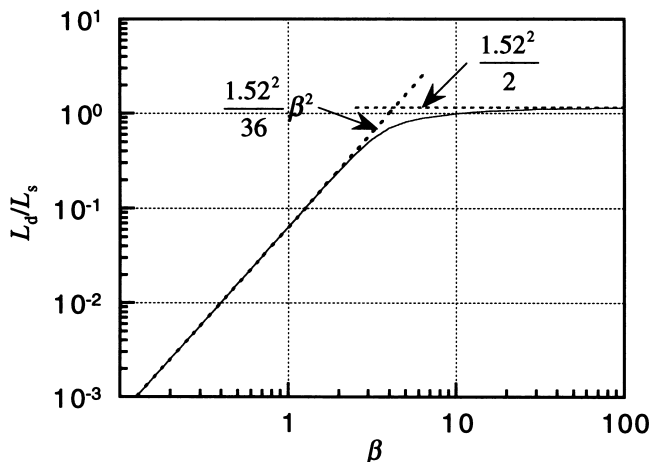


Figure 8 Thermally developing length and its asymptotes

$$L_d/L_s = \frac{0.232}{\pi} \beta^2. \quad (38)$$

The above equation is identical with that for the case of $\beta \rightarrow 0$ in Equation (36) except the proportional constant. This fact states that the basic heat transfer characteristics of slow oscillating flow are almost the same as that of the unidirectional steady flow. Meanwhile, the coefficient in Equation (38) is larger by 15% than that in Equation (36), which implies that the true length of the thermally developing region by slow oscillating flow is smaller by 15% than that obtained by quasi-steady assumption. This discrepancy comes mainly from the difference in the wall boundary conditions. In case of oscillating flow, as discussed earlier, the heat flux at the inner wall oscillates due to the effect of thermal capacity of the wall, while the period-averaged heat flux is kept constant. Therefore, the heat transfer characteristics of the oscillating flow cannot be simulated exactly by the quasi-steady assumption presuming the wall heat flux kept constant every instant. Particularly, in case of heat flux forcing, the instantaneous heat flux at the inner wall calculated from Equation (14) is not constant in time and not even distributed in a square-wave form over the axial direction.

Heat transfer in the thermally developing region

As discussed earlier, the Nusselt number in the fully developed region is a constant equal to Nu_0 . But in the thermally developing region, the Nusselt number is a function of the axial position because the temperature difference varies along the axis while the wall heat flux is constant.

To investigate the effect of the thermally developing region on the heat transfer, the local average Nusselt number, $\tilde{Nu}_z$, is defined as

$$\tilde{Nu}_z = 2 \int_0^z \bar{q}_w dz / \int_0^z \langle \Delta \theta \rangle dz. \quad (39)$$

The introduction of the integrated average inherits from the need of estimating the performance of a heat exchanger when the available length is specified. Given the length of a heat exchanger by z , the overall average of the Nusselt number for the heat exchanger is estimated to be $\tilde{Nu}_z$.

Figure 9 shows the local average Nusselt number defined in Equation (39) for several values of β . The horizontal axis is normalized with the thermally developing length, while the vertical axis is normalized with the local Nusselt number in the thermally fully developed region, i.e. Nu_0 . It is noticeable that the curves make a narrow band in the figure, which means that the effect of β on the overall heat transfer does not vary significantly if the thermally developing length is selected as a characteristic length scale. It can be also seen that the local average Nusselt number is approximately proportional to $(z/L_d)^{-1/2}$ in the developing region and approaches Nu_0 as z/L_d becomes greater than 1. From the design standpoint, for the heat exchanger longer than the thermally developing region, the heat transfer is enhanced only in the developing region and remains saturated out of the developing region. Therefore, the augmentation of the heat transfer rate can be achieved

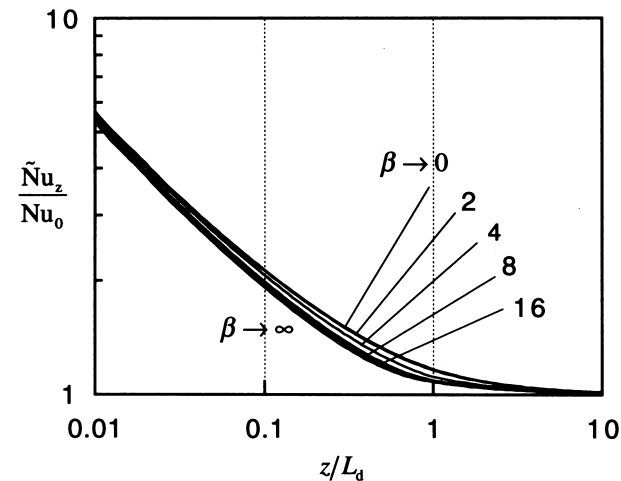


Figure 9 Local average Nusselt number in the thermally developing region

by reducing the length of the heat exchanger, at least shorter than the thermally developing length. Nevertheless, there must be a trade-off between the heat transfer enhancement and the decrease in the heat transfer area.

Conclusion

The heat transfer characteristics of an oscillating flow in a tube are investigated based on a theoretical model of a heat exchanger comprising a heater and a cooler. The present model admits the presence of a thermal discontinuity along the axis and therefore is more realistic than the previous studies dealing with linear temperature distributions. Two types of thermal forcings (i.e. either the temperature or the heat flux has a square-wave distribution) are considered as the tube wall condition. The solutions are obtained by superposing the particular solution pertaining to sinusoidal temperature distributions. The results are analyzed and presented focusing on the thermally developing length and the Nusselt number variation.

For the temperature forcing, the thermal developing length approaches an asymptotic value which is equal to the swept distance of the working fluid.

The heat flux forcing better simulates the thermal condition of a heat exchanger and therefore is analyzed in more detail. For the heat flux forcing, the thermal developing length increases in proportion to the oscillation frequency (i.e. β^2) at low frequency. The participating proportional constant is deduced from the similarity of the oscillating flow to the unidirectional flow at a low frequency limit. However, at a high frequency limit, the thermally developing length assumes an asymptotic value of an order of the swept distance. Within the thermally developing region, the local Nu is inversely proportional to the square root of the distance from the thermal discontinuity. On the other hand, over the fully developed region, the heat transfer rate is almost saturated and is identical to that corresponding to a sinusoidal temperature distribution with an infinitesimally small swept length ratio. In light of the heat exchanger design, this confirms that the length of the heat exchanger needs to be order of the swept distance.

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References

1. Kurzweg, U.H., *J. Fluid Mech.*, 1985, **156**, 291.
2. Kurzweg, U.H., *Int. J. Heat Mass Transfer*, 1986, **29**, 1969.
3. Gedeon, D., *ASME J. Heat Transfer*, 1986, **108**, 513.
4. Swift, G.W., *J. Acoust. Soc. Am.*, 1988, **84**, 1145.
5. Tominaga, A., *Cryogenics*, 1995, **35**, 427.
6. Simon, T.W. and Seume, J.R. A survey of oscillating flow in Stirling engine heat exchangers. NASA CR-182108 (1988).
7. Atebek, H.B. and Chang, C.C., *Z. angew. Math. Phys.*, 1961, **12**, 185.
8. Iguchi, M., Park, G.M., Akao, F. and Yamamoto, F., *JSME Int. J.*, 1992, **35**, 158.
9. Siegel, R. and Perlmutter, M., *ASME J. Heat Transfer*, 1962, 111.
10. Kim, S.Y., Kang, B.H. and Hyun, J.M., *Int. J. Heat Mass Transfer*, 1993, **36**, 4257.
11. Lee, D.-Y., Park, S.-J. and Ro, S.T., *Int. J. Heat Mass Transfer*, 1995, **38**, 2529.
12. Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integrals, Series, and Products: Corrected and Enlarged Edition*. Academic Press, New York, 1980.
13. Bejan, A., *Convective Heat Transfer*, Chap. 3. Wiley, New York, 1984.